Exam Review Solutions

(1) Calculate the following:

$$\frac{d}{dx}\int_1^{\sqrt{x}} \left[\frac{d}{dt}\int_1^{t^2} \frac{\sin s}{s}\, ds \right] dt$$

First, we have

$$\frac{d}{dt}\int_1^{t^2} \frac{\sin s}{s}\, ds = \frac{\sin(t^2)}{t^2} \cdot \frac{d}{dt}(t^2)$$

$$= \frac{\sin(t^2)}{t^2} \cdot 2t$$

$$= \frac{2\sin(t^2)}{t}.$$

Plugging back in,

$$\frac{d}{dx}\int_1^{\sqrt{x}} \frac{2\sin(t^2)}{t}\, dt = \frac{2\sin(x)}{\sqrt{x}} \cdot \frac{d}{dx}(\sqrt{x})$$

$$= \frac{2\sin(x)}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}$$

$$= \boxed{\frac{\sin(x)}{x}}$$

(2) Calculate

$$\int (x+1)(x+3)^{2016}\, dx$$

Let $u = x+3$, then $du = dx$ and

$x = u-3$. So, $x+1 = u-2$.

Then, $$\int (x+1)(x+3)^{2016}\, dx = \int (u-2)(u)^{2016}\, du$$

$$= \int u^{2017} - 2u^{2016}\, du$$

$$= \frac{u^{2018}}{2018} - \frac{2u^{2017}}{2017} + C$$

$$= \boxed{\frac{(x+3)^{2018}}{2018} - \frac{2(x+3)^{2017}}{2017} + C}$$

3) Evaluate the following integrals

$$\int_1^e \frac{(\ln(x))^2}{x}\,dx$$

Let $u = \ln x$ Then, $du = \frac{1}{x} \cdot dx$. So, $dx = x\,du$.

Moreover, our bounds become $\ln(1) = 0$ to $\ln(e) = 1$.

So, $$\int_1^e \frac{(\ln(x))^2}{x}\,dx = \int_0^1 u^2\,du = \frac{u^3}{3}\Big|_0^1 = \boxed{\frac{1}{3}}$$

(4) Find the flow rate through a tube of radius 4cm, assuming that the velocity of fluid particles at a distance r centimeter from the center is $v(r) = (16 - r^2)$ cm/s.

Recall Flow rate $Q = 2\pi \int_0^R r v(r)\, dr$.

Hence, we have $Q = 2\pi \int_0^4 r(16 - r^2)\, dr$

$$= 2\pi \int_0^4 16r - r^3\, dr$$

$$= 2\pi \left(\frac{16r^2}{2} - \frac{r^4}{4} \right) \Big|_0^4$$

$$= 2\pi \left(8(4)^2 - \frac{(4)^4}{4} \right)$$

$$= \pi (16^2 - 16 \cdot 8)$$

$$= \pi (256 - 128)$$

$$= \boxed{\pi \cdot 128\ cm^3/s}$$

) (a) Let m be the average value of $f(x) = x^4$ on $[0,3]$. Find a value c in $[0,3]$ such that $f(c) = m$.

Recall that the average value of $f(x)$ on $[a,b]$ is given by

$$\text{Average Value} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

So, $$m = \frac{1}{3-0}\int_0^3 x^4\,dx = \frac{1}{3}\left(\frac{x^5}{5}\Big|_0^3\right) = \frac{3^5}{3\cdot 5} = \frac{3^4}{5}.$$

We want $f(c) = \frac{3^4}{5} \Rightarrow c^4 = \frac{3^4}{5}$.

So, $\boxed{c = \frac{3}{5^{1/4}}}$

5 (b). Let $f(x) = \sqrt{x}$. Find a value c in $[4,9]$ such that $f(c)$ is equal to the average of f on $[4,9]$.

We have

$$\text{Average} = \frac{1}{9-4}\int_4^9 x^{1/2}\,dx$$

$$= \frac{1}{5}\left(\frac{2}{3}\cdot x^{3/2}\right)\Big|_4^9$$

$$= \frac{2}{15}\left[3^3 - 2^3\right]$$

$$= \frac{2}{15}(27-8)$$

$$= \frac{2\cdot 19}{15}$$

$$= \frac{38}{15}.$$

So, $f(c) = \frac{38}{15} \Rightarrow \sqrt{c} = \frac{38}{15} \Rightarrow c = \left(\frac{38}{15}\right)^2$.

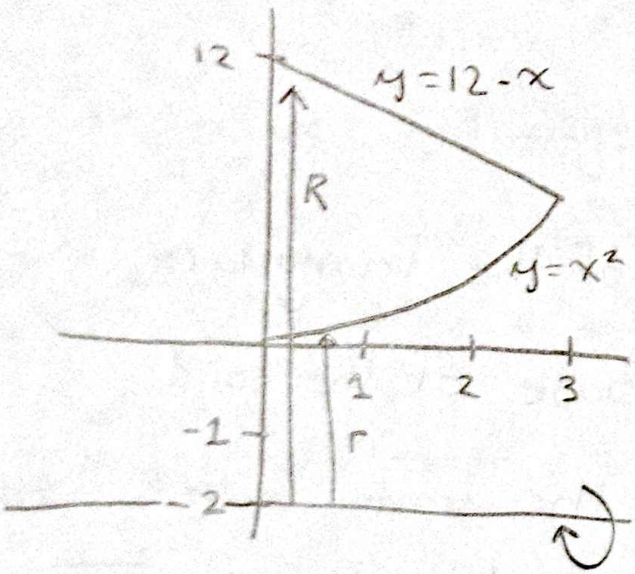
6) Find the volume of the solid whose base is the unit circle $x^2+y^2=1$, and the cross sections perpendicular to the x-axis are triangles whose height and base are equal.

At each location x, the side of the triangular cross section that lies in the base of the solid extends from the top half of the unit circle (with $y=\sqrt{1-x^2}$) to the bottom half (with $y=-\sqrt{1-x^2}$). The triangle therefore has base and height $2\sqrt{1-x^2}$ and so its area is $\frac{1}{2}b \cdot h \Rightarrow$

$A(x)=2(1-x^2)$. So, the volume, V, is

$$V=\int_{-1}^{1} 2(1-x^2)\,dx = 2x-\frac{2x^3}{3}\Big|_{-1}^{1} = \frac{8}{3}.$$

(7) Find the volume of the solid obtained by rotating the region enclosed by the graphs $y = x^2$, $y = 12 - x$, $x = 0$, about $y = -2$.



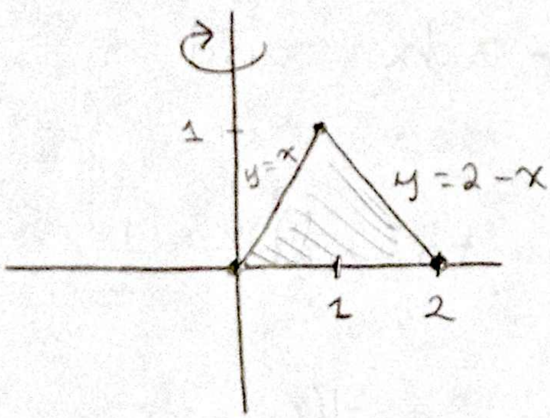
$$V = \pi \int_0^3 R^2 - r^2 \, dx = \pi \int_0^3 (12 - x + 2)^2 - (x^2 + 2)^2 \, dx$$

$$= \pi \int_0^3 (14 - x)^2 - (x^2 + 2)^2 \, dx$$

$$= \pi \int_0^3 192 - 28x - 3x^2 - x^4 \, dx = \frac{1872\pi}{5}$$

Use the shell method to compute the volume obtained by rotating the region enclosed by the graphs $y = 1 - |x-1|$, $y = 0$ about the y-axis.

$$1 - |x-1| = \begin{cases} x & \text{if } x \in [0,1] \\ 2-x & \text{if } x \in [1,2] \end{cases}$$



When rotating this region about the y-axis two different shells are generated. For each $x \in [0,1]$, the shell has radius x and height x; for each $x \in [1,2]$ the shell has radius x and height $2-x$. The volume of the resulting solid is

$$V = 2\pi \int_0^1 x^2\,dx + 2\pi \int_1^2 x(2-x)\,dx = 2\pi\left(\frac{1}{3}x^3\right)\Big|_0^1 + 2\pi\left(x^2 - \frac{1}{3}x^3\right)\Big|_1^2$$

$$= 2\pi.$$

(9) (a) Calculate $\int_0^1 xe^{-x^2/2}\,dx$.

(b) $\int e^x \cos(e^x)\,dx$.

(a) Let $u = -\frac{x^2}{2}$. Then, $du = -x\,dx$ and

our bounds change; $-\frac{0^2}{2} = 0$ to $\frac{-(1)^2}{2} = -\frac{1}{2}$.

So, $\int_0^1 xe^{-x^2/2}\,dx = -\int_0^{-1/2} e^u\,du = -e^u \Big|_0^{-1/2}$

$= -e^{-1/2} + 1$

$= 1 - e^{-1/2}$.

(b) Let $u = e^x$. Then, $du = e^x\,dx$. So,

$\int e^x \cos(e^x)\,dx = \int \cos(u)\,du = \sin(u) + C$

$= \sin(e^x) + C$

Figure 2 shows a solid whose horizontal cross section at height y is a circle of radius $(1+y)^{-2}$ for $0 \le y \le H$. Find the volume of the solid as a function of H. What is the volume when $H=1$?

The area of each horizontal cross section is $A(y) = \pi(1+y)^{-4}$. Therefore, the volume of the solid is

$$\int_0^H \pi(1+y)^{-4}\,dy = \frac{\pi(1+y)^{-3}}{-3}\Bigg|_0^H = \pi\left(\frac{(1+H)^{-3}}{-3} + \frac{1}{3}\right)$$

$$= \frac{\pi}{3}\left(1 - \frac{1}{(1+H)^3}\right).$$

When $H=1$, the volume is $\frac{\pi}{3}\left(1 - \frac{1}{2^3}\right)$

$$= \frac{\pi}{3}\left(\frac{7}{8}\right) = \frac{7\pi}{24}.$$

(11) A tank of mass 20 kg containing 100 kg of water (density 1,000 kg/m³) is raised vertically at a constant speed of 100 m/min for 1 min, during which time water is leaking at a rate of 40 kg/min. Calculate the total work performed in raising the container.

Let t denote the elapsed time in minutes and let y denote the height of the container. Given that the speed of ascent is $100 \, m/min$, $y = 100t$;

moreover the mass of water in the container is

$$100 - 40t = 100 - 0.4y \text{ kg.}$$

The force needed to lift the container and its contents is then

$$9.8(20 + (100 - 0.4y)) = 1176 - 3.92y \text{ N},$$

and the work required to lift the container ↳

nd its contents is

$$\int_0^{100} (1176 - 3.92y)\,dy = (1176y - 1.96y^2)\Big|_0^{100} = 98,000\text{J}.$$

12. Water is pumped into a spherical tank of radius 2m from a source located 1m below a hole at the bottom (Fig. 5). The density of water is 1,000 kg/m³.

Calculate the work F(h) required to fill the tank to level h meters in the sphere.

Place the origin at the base of the sphere with the positive y-axis pointing upward. The equation of the great circle of the sphere is then $x^2 + (y-2)^2 = 4$. At location y, the horizontal cross section is a circle of radius $\sqrt{4-(y-2)^2} = \sqrt{4y-y^2}$; $\hookrightarrow$

the volume of the layer is then

$\pi(4y - y^2)\Delta y \; m^3$. So, the force needed to

lift the layer is $1000(9.8)\pi(4y-y^2)\Delta y$ N.

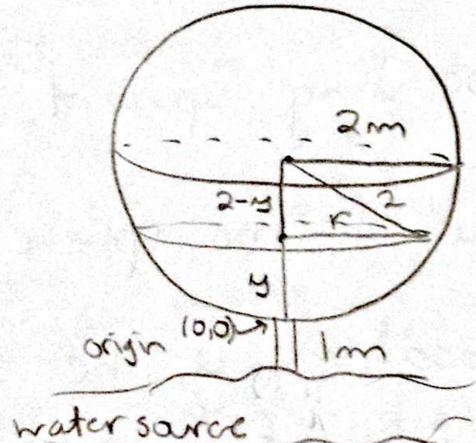
The layer must be lifted $y+1$ meters.

So, the work required to fill the tank is

$$9800\pi\int_0^h (y+1)(4y-y^2)\,dy = 9800\pi\int_0^h 3y^2+4y-y^3\,dy$$

$$= 9800\pi\left(y^3+2y^2-\frac{1}{4}y^4\right)\Big|_0^h$$

$$= 9800\pi\left(h^3+2h^2-\frac{1}{4}h^4\right) \text{ J.}$$



$r = \sqrt{2^2-(2-y)^2}$

$= \sqrt{4-(2-y)^2}$

So, $A(y) = \pi r^2$

$= \pi(4-(2-y)^2)$

$= \pi(4-(y-2)^2)$